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On the Denseness of Rational Systems

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This note characterizes the denseness of rational systems

$$\mathscr{P}_{n-1}(a_1, ..., a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^{n} (x-a_k)}, P \in \mathscr{P}_{n-1} \right\} \qquad (n=1, 2, ...),$$

in $C[-1, 1]$, where the nonreal poles in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus [-1, 1]$ are paired by complex conjugation. This extends an Achiezer's result. © 1999 Academic Press

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1. INTRODUCTION

We let

$$\mathscr{P}_m(a_1; ..., a_n) := \left\{ \frac{P(x)}{\sum_{k=1}^{n} |x-a_k|}, P \in \mathscr{P}_m \right\} \tag{1.1}$$

with $\{a_k\}_{k=1}^{n} \subset \mathbb{C} \setminus [-1, 1]$, where $\mathscr{P}_m$ is the set of all real algebraic polynomials of degree at most m. It is easy to see that $\mathscr{P}_m(a_1, ..., a_n)$ is a linear space and $\mathscr{P}_m(a_1, ..., a_n) \subset \mathscr{P}_M(a_1, ..., a_n)$ for $m < M$. We define the numbers $\{c_k\}_{k=1}^{n}$ by

$$a_k := \frac{c_k + c_k^{-1}}{2}, \qquad |c_k| < 1. \tag{1.2}$$

When all the poles $\{a_k\}_{k=1}^{n}$ are real and distinct, $\mathscr{P}_{n-1}(a_1, a_2, ..., a_n)$ is simply the real span of the following system

$$\left\{ \frac{1}{x-a_1}, \frac{1}{x-a_2}, ..., \frac{1}{x-a_n} \right\}, \qquad x \in [-1, 1]. \tag{1.3}$$

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With respect to the denseness of $\operatorname{span}\{1/(x-a_k)\}_{k=1}^{\infty}$, the following well-known result is due to Achiezer [1, Problem 7, p. 254]:

Achiezer Theorem. *Let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$ be distinct. Then $\operatorname{span}\{1/(x-a_k)\}_{k=1}^{\infty}$ is dense in $C[-1, 1]$ if and only if*

$$\sum_{k=1}^{\infty} (1-|c_k|) = \infty.$$

Recently, Borwein and Erdélyi [3] also proved this by using entirely different methods.

Note that $\mathscr{P}_{n-1}(a_1, ..., a_n)$ is still a real rational space when the nonreal poles form complex conjugate pairs, moreover, $\prod_{k=1}^{n} |x-a_k|$ can be replaced by $\prod_{k=1}^{n} (x-a_k)$. So, it is natural to ask whether we can extend Achiezer's Theorem to the case: the repeated poles are allowed and the nonreal elements in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus [-1, 1]$ are paired by complex conjugation.

In this note, we consider this question and give an affirmative answer. More precisely, we have

Theorem 1.1. *Let the nonreal elements in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus [-1, 1]$ be paired by complex conjugation. Then $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$ are dense in $C[-1, 1]$ if and only if*

$$\sum_{k=1}^{\infty} (1-|c_k|) = \infty. \tag{1.4}$$

2. PROOF OF THEOREM 1.1

Our proof of Theorem 1.1 is mainly based on the Chebyshev polynomials with respect to $\mathscr{P}_n(a_1, ..., a_n)$ constructed recently by Borwein, Erdélyi, and Zhang [4]. The explicit formulae for the Chebyshev polynomials for the system $\mathscr{P}_n(a_1, a_2, ..., a_n)$ are implicitly contained in Achiezer [1] provided that $\{a_k\}_{k=1}^{n} \subset \mathbb{R} \setminus [-1, 1]$ are distinct. It should be mentioned that they [4] allow *repeated poles* and *nonreal poles* in this system, in which the nonreal poles form complex conjugate pairs (cf. [4]). We use $T_n(x)$ to denote the Chebyshev polynomial of the first kind with respect to $\mathscr{P}_n(a_1, a_2, ..., a_n)$. For convenience, we include its construction here.

Let

$$M_n(z) := \left(\prod_{k=1}^{n} (z-c_k)(z-\bar{c}_k) \right)^{1/2}, \tag{2.1}$$

where the square root is defined so that $M_n^*(z) = z^n M_n(z^{-1})$ is analytic in a neighbourhood of the closed unit disk, and let

$$f_n(z) := \frac{M_n(z)}{z^n M_n(z^{-1})}. \tag{2.2}$$

Then the *Chebyshev polynomial of the first kind* for the rational space $\mathscr{P}_n(a_1, a_2, ..., a_n)$ is defined by

$$T_n(x) := \frac{f_n(z) + 1/f_n(z)}{2}, \qquad x = \frac{z + z^{-1}}{2}, \qquad |z| = 1. \tag{2.3}$$

In fact $T_n(x)$ is a rational function. More precisely, we conclude that $T_n \in \mathscr{P}_n(a_1, ..., a_n)$ (cf. [4, Theorem 1.2]). It is shown [4] that these Chebyshev polynomials preserve almost all the elementary properties of the classical Chebyshev polynomials.

LEMMA 2.1. *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C} \backslash [-1, 1]$ be paired by complex conjugation, and let T_n be the Chebyshev polynomial of the first kind associated with $\mathscr{P}_n(a_1, ..., a_n)$. Then the best approximation to* 1 *from $\mathscr{P}_{n-1}(a_1, a_2, ..., a_n)$ is*

$$p := 1 - T_n/A_0. \tag{2.4}$$

Moreover, we have

$$\|1 - p\|_{[-1,1]} = 1/|A_0|, \tag{2.5}$$

where A_0 is the constant term in T_n:

$$A_0 := \frac{(-1)^n}{2}\left((c_1 \cdots c_n)^{-1} + c_1 \cdots c_n\right). \tag{2.6}$$

Proof. Clearly, there exists some $r \in \mathscr{P}_{n-1}(a_1, ..., a_n)$ such that

$$T_n(x) := A_0 + r(x).$$

Then we conclude that

$$A_0 = \lim_{x \to \infty} T_n(x),$$

furthermore, by the construction of T_n, it is easy to show (2.6) (cf. [4, Proposition 4.1]). The conclusions of (2.4) and (2.5) can be proved by the same fashion as Lemma 2.2, that is by the counting zeros' argument. We omit it. ∎

Let $a \in \mathbb{R} \backslash [-1, 1]$ such that $a \notin \{a_k\}_{k=1}^n$, then we define the constant c such that

$$a = \frac{c + c^{-1}}{2}, \qquad |c| < 1. \tag{2.7}$$

Let T_{n+1} be the Chebyshev polynomial of the first kind with respect to $\mathscr{P}_{n+1}(a_1, ..., a_n, a)$. Lemma 2.2 gives the best approximation to $1/x - a$ from $\mathscr{P}_n(a_1, ..., a_n)$.

LEMMA 2.3. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \backslash [-1, 1]$ *be paired by complex conjugation. Then, for* $a \in \mathbb{R} \backslash [-1, 1]$ *and* $a \notin \{a_k\}_{k=1}^n$, *the best approximation to* $1/x - a$ *from* $\mathscr{P}_n(a_1, ..., a_n)$ *on* $[-1, 1]$ *is*

$$q := \frac{1}{x - a} - \frac{T_{n+1}(x)}{B_{n+1}} \tag{2.8}$$

and

$$\left\| \frac{1}{x - a} - q(x) \right\|_{[-1, 1]} = \frac{1}{|B_{n+1}|}, \tag{2.9}$$

where

$$B_{n+1} := -\left(\frac{c - c^{-1}}{2} \right)^2 \prod_{j=1}^{n} \frac{1 - c c_j}{c - c_j}. \tag{2.10}$$

Proof. We prove it by the counting zeros' argument. Since $a \in \mathbb{R} \backslash [-1, 1]$ and $a \notin \{a_k\}_{k=1}^n$, we then can construct the Chebyshev polynomial of the first kind T_{n+1} for $\mathscr{P}_{n+1}(a_1, ..., a_n, a)$ and it can be expressed as

$$T_{n+1}(x) := s(x) + \frac{B_{n+1}}{x - a},$$

where $r \in \mathscr{P}_n(a_1, ..., a_n)$. Since

$$B_{n+1} = \lim_{x \to a} (x - a)\, T_{n+1}(x),$$

then it is easy to show (2.10) by a simple calculation. Moreover, $q(x) = -s(x)/B_{n+1}$. Note that (cf. [4, Theorem 1.2]) $\|T_{n+1}\|_{[-1, 1]} = 1$, we have

$$\left\| \frac{1}{x - a} - q(x) \right\|_{[-1, 1]} = \frac{1}{|B_{n+1}|}. \tag{2.11}$$

If there exists some $t \in \mathscr{P}_n(a_1, ..., a_n)$ such that

$$\left\| \frac{1}{x-a} - t(x) \right\|_{[-1,1]} < \frac{1}{|B_{n+1}|}, \tag{2.12}$$

recall that (cf. [4, Theorem 1.2]) there exist $n+2$ nodes: $-1 = y_{n+1} < y_n < \cdots < y_1 < y_0 = 1$ such that $T_{n+1}(y_j) = (-1)^j$, $j = 0, ..., n, n+1$. Hence,

$$\frac{T_{n+1}}{B_{n+1}} - \left(\frac{1}{x-a} - t(x) \right) = -q + t \in \mathscr{P}_n(a_1, ..., a_n)$$

changes sign between any two consecutive extreme points of T_{n+1}. Furthermore, $t - q$ has at least $n+1$ zeros in $(-1, 1)$ and consequently, it must vanish identically. This contradicts (2.12). ∎

Proof of Theorem 1.1. We first prove *only if* part. Note that $|c_k| < 1$ $(k = 1, 2, ...)$ and by (2.6) we then have

$$\prod_{k=1}^{n} |c_k| < \frac{1}{|A_0|} = \frac{2 \prod_{k=1}^{n} |c_k|}{1 + \prod_{k=1}^{n} |c_k|^2} \leqslant 2 \prod_{k=1}^{n} |c_k|.$$

If $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$ are dense in $C[-1, 1]$, then by Lemma 2.1 we have $1/|A_0| \to 0 (n \to \infty)$, that is, $\prod_{k=1}^{\infty} |c_k| = 0$, this is equivalent to (1.4).

Next we prove *if* part. By (2.10) we have

$$\frac{1}{|B_{n+1}|} \to \left(\frac{2}{c - c^{-1}} \right)^2 \prod_{j=1}^{\infty} \left| \frac{c - c_j}{1 - cc_j} \right| \qquad (n \to \infty).$$

Recall that $\prod_{k=1}^{\infty} (c - c_j)/(1 - cc_j)$ is an infinite Blaschke product. Then by [6, Theorem 1, p. 281] or [5, Theorem 15.23, p. 311] we conclude that (1.4) implies

$$\prod_{j=1}^{\infty} \left| \frac{c - c_j}{1 - cc_j} \right| = 0,$$

consequently, combining (2.9) we see that $1/(x-a)$ can be uniformly approximated in $\{\mathscr{P}_n(a_1, ..., a_n)\}$ on $[-1, 1]$ for $a \in \mathbb{R} \backslash [-1, 1]$. Also, if (1.4) holds, then from the proof of *only if* part and Lemma 2.1, we see that any constant can be uniformly approximated in $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$. Note that every function $R \in \mathscr{P}_n(a_1, ..., a_n)$ can be written in the form

$$R(x) = b_n + R_0(x), \qquad b_n \in \mathbb{R}, \qquad R_0 \in \mathscr{P}_{n-1}(a_1, ..., a_n),$$

and $\mathscr{P}_{n-1}(a_1, ..., a_n) \subset \mathscr{P}_{N-1}(a_1, ..., a_N)$ for $n < N$. Thus, (1.4) implies that $1/(x-a)$ can be uniformly approximated in $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$ on $[-1, 1]$. Note that $a \in \mathbb{R} \backslash [-1, 1]$ is an arbitrary number, so we can take a to be any of a sequence of distinct number such that they satisfy the condition (1.4), that mean $1/(x-a)$ can be taken as any of a dense sequence of distinct basis functions. Therefore, *if* part follows. ∎

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REFERENCES

1. N. I. Achiezer, "Theory of Approximation," Ungar, New York, 1956.
2. P. Borwein and T. Erdélyi, "Polynomials and Polynomial Inequalities," Springer-Verlag, New York, 1995.
3. P. Borwein and T. Erdélyi, Dense Markov space and unbounded Bernstein inequalities, *J. Approx. Theory* **81** (1995), 66–77.
4. P. Borwein, T. Erdélyi, and J. Zhang, Chebyshev polynomials and Markov–Bernstein type inequalities for the rational spaces, *J. London Math. Soc.* **50** (1994), 501–519.
5. W. Rudin, "Real and Complex Analysis," Third Edition, McGraw–Hill, New York, 1987.
6. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," Fourth Ed., Amer. Math. Soc. Coll. Publ., Vol. 20, Am. Math. Soc., Providence, 1965.